

# RAINBOW DOMINATION AND RELATED PROBLEMS ON SOME CLASSES OF PERFECT GRAPHS

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**Abstract.** Let  $k \in \mathbb{N}$  and let  $G$  be a graph. A function  $f : V(G) \rightarrow 2^{[k]}$  is a rainbow function if, for every vertex  $x$  with  $f(x) = \emptyset$ ,  $f(N(x)) = [k]$ . The rainbow domination number  $\gamma_{kr}(G)$  is the minimum of  $\sum_{x \in V(G)} |f(x)|$  over all rainbow functions. We investigate the rainbow domination problem for some classes of perfect graphs.

## 1 Introduction

Brešar et al. introduced the rainbow domination problem in 2008 [3].

The  $k$ -rainbow domination-problem drew our attention because it is solvable in polynomial time for classes of graphs of bounded rankwidth but, unless one fixes  $k$  as a constant, it seems not formulatable in monadic second-order logic.

Let us start with the definition.

**Definition 1.** Let  $k \in \mathbb{N}$  and let  $G$  be a graph. A function  $f : V(G) \rightarrow 2^{[k]}$  is a  $k$ -rainbow function if, for every  $x \in V(G)$ ,

$$f(x) = \emptyset \quad \text{implies} \quad \cup_{y \in N(x)} f(y) = [k]. \quad (1)$$

The  $k$ -rainbow domination number of  $G$  is

$$\gamma_{rk}(G) = \min \{ \|f\| \mid f \text{ is a } k\text{-rainbow function for } G \},$$

$$\text{where } \|f\| = \sum_{x \in V(G)} |f(x)|. \quad (2)$$

We call  $\|f\|$  the cost of  $f$  over the graph  $G$ . When there is danger of confusion, we write  $\|f\|_G$  instead of  $\|f\|$ . We call the elements of  $[k]$  the colors of the rainbow and, for a vertex  $x$  we call  $f(x)$  the label of  $x$ . For a set  $S$  of vertices we write

$$f(S) = \cup_{x \in S} f(x).$$

It is a common phenomenon that the introduction of a new domination variant is followed chop-chop by an explosion of research results and their write-ups.

One reason for the popularity of domination problems is the wide range of applicability and directions of possible research. We moved our bibliography of recent publications on this specific domination variant to the appendix. We refer to [25] for the description of an application of rainbow domination.

To begin with, Brešar et al. showed that, for any graph  $G$ ,

$$\gamma_{rk}(G) = \gamma(G \square K_k), \quad (3)$$

where  $\gamma$  denotes the domination number and where  $\square$  denotes the Cartesian product. This observation, together with Vizing's conjecture, stimulated the search for graphs for which  $\gamma = \gamma_{r2}$  (see also [1,15]). Notice that, by (3) and Vizing's upperbound  $\gamma_{rk}(G) \leq k \cdot \gamma(G)$  [23].

Chang et al. [6] were quick on the uptake and showed that, for  $k \in \mathbb{N}$ , the  $k$ -rainbow domination problem is NP-complete, even when restricted to chordal graphs or bipartite graphs. The same paper shows that there is a linear-time algorithm to determine the parameter on trees. A similar algorithm for trees appears in [32] and this paper also shows that the problem remains NP-complete on planar graphs.

Notice that (3) shows that  $\gamma_{rk}(G)$  is a non-decreasing function in  $k$ . Chang et al. show that, for all graphs  $G$  with  $n$  vertices and all  $k \in \mathbb{N}$ ,

$$\min \{ k, n \} \leq \gamma_{rk}(G) \leq n \quad \text{and} \quad \gamma_{rn}(G) = n. \quad (4)$$

For trees  $T$ , Chang et al. [6] give sharp bounds for the smallest  $k$  satisfying  $\gamma_{rk}(T) = |V(T)|$ .

Many other papers establish bounds and relations, eg, between the 2-rainbow domination number and the total domination number or the (weak) roman domination number [7,10,14,28,29], or study edge- or vertex critical graphs with respect to rainbow domination [18], or obtain results for special graphs such as paths, cycles, graphs with given radius, and the generalized Petersen graphs [2,11,21,23,24,26,27,30,31].

Pai and Chiu develop an exact algorithm and a heuristic for 3-rainbow domination. In [17] they present the results of some experiments. Let us mention that the  $k$ -rainbow domination number may be computed, via (3), by an exact, exponential algorithm that computes the domination number. For example, this shows that the  $k$ -rainbow domination number can be computed in  $O(1.4969^{nk})$  [21,22].

A  $k$ -rainbow family is a set of  $k$ -rainbow functions which sum to at most  $k$  for each vertex. The  $k$ -rainbow domatic number is defined as the maximal number of elements in such a family. Some results were obtained in [12,16,22].

Whenever domination problems are under investigation, the class of strongly chordal graphs are of interest from a computational point of view. Farber showed

that a minimum weight dominating set can be computed in polynomial time on strongly chordal graphs [11]. Recently, Chang et al. showed that the  $k$ -rainbow dominating number is equal to the so-called weak  $\{k\}$ -domination number for strongly chordal graphs [3,4,5]. A weak  $\{k\}$ -dominating function is a function  $g : V(G) \rightarrow \{0, \dots, k\}$  such that, for every vertex  $x$ ,

$$g(x) = 0 \quad \text{implies} \quad \sum_{y \in N(x)} g(y) \geq k. \quad (5)$$

The weak domination number  $\gamma_{wk}(G)$  minimizes  $\sum_{x \in V(G)} g(x)$ , over all weak  $\{k\}$ -dominating functions  $g$ . In their paper, Chang et al. show that the  $k$ -rainbow domination number is polynomial for block graphs. As far as we know, the  $k$ -rainbow domination number is open for strongly chordal graphs.

It is easy to see that, for each  $k$ , the  $k$ -rainbow domination problem can be formulated in monadic second-order logic. This shows that, for each  $k$ , the parameter is computable in linear time for graphs of bounded treewidth or rankwidth [9].

**Theorem 1.** *Let  $k \in \mathbb{N}$ . There exists a linear-time algorithm that computes  $\gamma_{rk}(G)$  for graphs of bounded rankwidth.*

For example, Theorem 1 implies that, for each  $k$ ,  $\gamma_{rk}(G)$  is computable in polynomial time for distance-hereditary graphs, ie, the graphs of rankwidth 1. Also, graphs of bounded outerplanarity have bounded treewidth, which implies bounded rankwidth.

A direct application of the monadic second-order theory involves a constant which is an exponential function of  $k$ . In the following section we show that, often, this exponential factor can be avoided.

## 2 $k$ -Rainbow domination on cographs

Cographs are the graphs without an induced  $P_4$ . As a consequence, cographs are completely decomposable by series and parallel operations, that is, joins and unions [12]. In other words, a graph is a cograph if and only if every nontrivial, induced subgraph is disconnected or its complement is disconnected. Cographs have a rooted, binary decomposition tree, called a cotree, with internal nodes labeled as joins and unions [8].

For a graph  $G$  and  $k \in \mathbb{N}$ , let  $F(G, k)$  denote the set of  $k$ -rainbow functions on  $G$ . Furthermore, define

$$F^+(G, k) = \{ f \in F(G, k) \mid \forall_{x \in V(G)} f(x) \neq \emptyset \} \quad (6)$$

$$\text{and } F^-(G, k) = F(G, k) \setminus F^+(G, k). \quad (7)$$

**Theorem 2.** *There exists a linear-time algorithm to compute the  $k$ -rainbow domination number  $\gamma_{rk}(G)$  for cographs  $G$  and  $k \in \mathbb{N}$ .*

*Proof.* We describe a dynamic programming algorithm to compute the  $k$ -rainbow domination number. A minimizing  $k$ -rainbow function can be obtained by backtracking.

It is easy to determine the minimal cost of  $k$ -rainbow functions that have no empty set-labels. We therefore concentrate on those  $k$ -rainbow functions for which some labels are empty sets.

Let  $k \in \mathbb{N}$ . For a cograph  $H$  define

$$R^+(H) = \min \{ \|f\|_H \mid f \in F^+(H, k) \text{ and } f(V(H)) = [k] \}, \quad (8)$$

$$R^-(H) = \min \{ \|f\|_H \mid f \in F^-(H, k) \}. \quad (9)$$

Here, we adopt the convention that  $R^-(H) = \infty$  if  $F^-(H, k) = \emptyset$ .

Notice that

$$\boxed{R^+(H) = \max \{ |V(H)|, k \}.} \quad (10)$$

Assume that  $H$  is the union of two smaller cographs  $H_1$  and  $H_2$ . Then

$$R^-(H) = \min \{ R^-(H_1) + |V(H_2)|, R^-(H_2) + |V(H_1)|, R^-(H_1) + R^-(H_2) \}. \quad (11)$$

Now assume that  $H$  is the join of two smaller cographs,  $H_1$  and  $H_2$ . Then we have

$$R^-(H) = \min \{ R^+(H_1), R^+(H_2), R^-(H_1), R^-(H_2), 2k \}. \quad (12)$$

We prove the correctness of Equation (12) below.

Let  $f$  be a  $k$ -rainbow function from  $F^-(H, k)$  with minimum cost over  $H$ . Consider the following cases.

- (a)  $f(x) \neq \emptyset$  for all  $x \in V(H_1)$ . Then there is a vertex with an empty label in  $H_2$ . Let  $L_2 = \cup_{z \in V(H_2)} f(z)$ . Define an other  $k$ -rainbow function  $f'$  as follows. For an arbitrary vertex  $x \in V(H_1)$  let  $f'(x) = f(x) \cup L_2$ . For all other vertices  $y$  in  $H_1$  let  $f'(y) = f(y)$  and for all vertices  $z$  in  $H_2$  let  $f'(z) = \emptyset$ . Notice that  $f'(V(H_1)) = [k]$ . So,  $f'$  is a  $k$ -rainbow function with at most the same cost as  $f$ . This shows that  $R^-(H) = R^+(H_1)$ .
- (b)  $f(y) \neq \emptyset$  for all  $y \in V(H_2)$ . This case is similar to the previous, so that  $R^-(H) = R^+(H_2)$ .
- (c)  $f(x) = f(y) = \emptyset$  for some  $x \in V(H_1)$  and some  $y \in V(H_2)$ . Let

$$L_1 = f(V(H_1)) \quad \text{and} \quad L_2 = f(V(H_2)).$$

For each color  $\ell \in [k]$ , let  $\nu_\ell$  be the number of times that  $\ell$  is used as a label, that is,

$$\nu_\ell = |\{ x \mid x \in V(H) \text{ and } \ell \in f(x) \}|.$$

Consider the following two subcases.

- (i) There exists some  $\ell$  with  $\nu_\ell = 1$ . Let  $u$  be the unique vertex with  $\ell \in f(u)$ . Assume that  $u \in V(H_1)$ . The case where  $u \in V(H_2)$  is similar. Then,  $u$  is adjacent to all  $x \in V(H)$  with  $f(x) = \emptyset$ . Modify  $f$  to  $f'$ , such that  $f'(u) = f(u) \cup (L_2 \setminus L_1)$ ,  $f'(x) = f(x)$  for all  $x \in V(H_1) \setminus \{u\}$ , and  $f'(y) = \emptyset$  for all  $y \in V(H_2)$ . Then  $f'$  is a  $k$ -rainbow function from  $F^-(H, k)$  and the cost of  $f'$  is at most the cost of  $f$ . Moreover,  $f'$  restricted to  $H_1$  is a  $k$ -rainbow function with minimum cost over  $H_1$ . Thus, in this case,  $R^-(H) = R^-(H_1)$ .
- (ii) For all  $\ell$ ,  $\nu_\ell \geq 2$ . Then, the cost of  $f$  over  $H$  is at least  $2k$ . In this case, we use an alternative function  $f'$ , which selects a certain vertex  $u$  in  $H_1$  and a certain vertex  $v$  in  $H_2$ , and set  $f'(u) = f'(v) = [k]$ . For all vertices  $z \in V(H) \setminus \{u, v\}$ , let  $f'(z) = \emptyset$ . The cost of  $f'$  is  $2k$  (which is at most the cost of  $f$ ), and  $f'$  remains a  $k$ -rainbow function from  $F^-(H, k)$ . Thus, in this case,  $R^-(H) = 2k$ .

This proves the correctness of Equation (12).

At the root of the cotree, we obtain  $\gamma_{rk}(G)$  via

$$\gamma_{rk}(G) = \min \{ |V(G)|, R^-(G) \}. \quad (13)$$

The cotree can be obtained in linear time (see, eg, [5,7,15]). Each  $R^+(H)$  is obtained in  $O(1)$  time via Equation (10), and  $R^-(H)$  is obtained in  $O(1)$  time via Equations (11) and (12).

This proves the theorem.  $\square$

The weak  $\{k\}$ -domination number (recall the definition near Equation (5)) was introduced by Brešar, Henning and Rall in [4] as an accessible, ‘monochromatic version’ of  $k$ -rainbow domination. In the following theorem we turn the tables.

In general, for graphs  $G$  one has that  $\gamma_{wk}(G) \leq \gamma_{rk}(G)$  since, given a  $k$ -rainbow function  $f$  one obtains a weak  $\{k\}$ -dominating function  $g$  by defining, for  $x \in V(G)$ ,  $g(x) = |f(x)|$ . The parameters  $\gamma_{wk}$  and  $\gamma_{rk}$  do not always coincide. For example  $\gamma_{w2}(C_6) = 3$  and  $\gamma_{r2}(C_6) = 4$ . Brešar et al. ask, in their Question 3, for which graphs the equality  $\gamma_{w2}(G) = \gamma_{r2}(G)$  holds. As far as we know this problem is still open. Chang et al. showed that weak  $\{k\}$ -domination and  $k$ -rainbow domination are equivalent for strongly chordal graphs [5].

For cographs equality does not hold. For example,

$$\text{when } G = (P_3 \oplus P_3) \otimes (P_3 \oplus P_3) \quad \text{then} \quad \gamma_{w3}(G) = 4 \quad \text{and} \quad \gamma_{r3}(G) = 6. \quad (14)$$

Let  $G$  be a graph and let  $k \in \mathbb{N}$ . For a function  $g : V(G) \rightarrow \{0, \dots, k\}$  we write  $\|g\|_G = \sum_{x \in V(G)} g(x)$ . Furthermore, for  $S \subset V(G)$  we write  $g(S) = \sum_{x \in S} g(x)$ .

**Theorem 3.** *There exists an  $O(k^2 \cdot n)$  algorithm to compute the weak  $\{k\}$ -domination number for cographs when a cotree is a part of the input.*

*Proof.* Let  $k \in \mathbb{N}$ . For a cograph  $H$  and  $q \in \mathbb{N} \cup \{0\}$ , define

$$W(H, q) = \min \{ \|g\|_H \mid g : V(H) \rightarrow \{0, \dots, k\} \text{ and } \forall_{x \in V(G)} g(x) = 0 \Rightarrow g(N(x)) + q \geq k \}. \quad (15)$$

When a cograph  $H$  is the union of two smaller cographs  $H_1$  and  $H_2$  then

$$\gamma_{wk}(H) = \gamma_{wk}(H_1) + \gamma_{wk}(H_2). \quad (16)$$

In such a case, we have

$$W(H, q) = W(H_1, q) + W(H_2, q). \quad (17)$$

When a cograph  $H$  is the join of two cographs  $H_1$  and  $H_2$  then the minimal cost of a weak  $\{k\}$ -dominating function is bounded from above by  $2k$ . Then

$$W(H, q) = \min \{ W_1 + W_2 \mid W_1 = W(H_1, q + W_2) \text{ and } W_2 = W(H_2, q + W_1) \}. \quad (18)$$

The weak  $\{k\}$ -domination number of a cograph  $G$ ,  $W(G, 0)$ , can be obtained via the above recursion, spending  $O(k^2)$  time in each of the  $n$  nodes in the cotree. This completes the proof.  $\square$

*Remark 1.* A  $\{k\}$ -dominating function [8]  $g : V(G) \rightarrow \{0, \dots, k\}$  satisfies

$$\forall_{x \in V(G)} g(N[x]) \geq k.$$

The  $\{k\}$ -domination number  $\gamma_{\{k\}}(G)$  is the minimal cost of a  $\{k\}$ -dominating function. A similar proof as for Theorem 3 shows the following theorem.

**Theorem 4.** *There exists an  $O(k^2 \cdot n)$  algorithm to compute  $\gamma_{\{k\}}(G)$  when  $G$  is a cograph.*

Similar results can be obtained for, eg, the  $(j, k)$ -domination number, introduced by Rubalcaba and Slater [19,20].

*Remark 2.* A frequently studied generalization of cographs is the class of  $P_4$ -sparse graphs. A graph is  $P_4$ -sparse if every set of 5 vertices induces at most one  $P_4$  [16,18]. We show in Appendix A that the rainbow domination problem can be solved in linear time on  $P_4$ -sparse graphs.

### 3 Weak $\{k\}$ -L-domination on trivially perfect graphs

Chang et al. were able to solve the  $k$ -rainbow domination problem (and the weak  $\{k\}$ -domination problem) for two subclasses of strongly chordal graphs, namely for trees and for blockgraphs. In order to obtain linear-time algorithms, they introduced a variant, called the weak  $\{k\}$ -L-domination problem [5,6]. In this section we show that this problem can be solved in  $O(k \cdot n)$  time for trivially perfect graphs.

**Definition 2.** A  $\{k\}$ -assignment of a graph  $G$  is a map  $L$  from  $V(G)$  to ordered pairs of elements from  $\{0, \dots, k\}$ . Each vertex  $x$  is assigned a label  $L(x) = (a_x, b_x)$ , where  $a_x$  and  $b_x$  are elements of  $\{0, \dots, k\}$ . A weak  $\{k\}$ -L-dominating function is a function  $w : V(G) \rightarrow \{0, \dots, k\}$  such that, for each vertex  $x$  the following two conditions hold.

$$w(x) \geq a_x, \quad \text{and} \quad (19)$$

$$w(x) = 0 \quad \Rightarrow \quad w(N[x]) \geq b_x. \quad (20)$$

The weak  $\{k\}$ -L-domination number is defined as

$$\gamma_{wkL}(G) = \min \{ \|g\| \mid g \text{ is a weak } \{k\}\text{-L-dominating function on } G \}. \quad (21)$$

Notice that

$$\forall_{x \in V(G)} L(x) = (0, k) \quad \Rightarrow \quad \gamma_{wk}(G) = \gamma_{wkL}(G). \quad (22)$$

**Definition 3.** A graph is trivially perfect if it has no induced  $P_4$  or  $C_4$ .

Wolk investigated the trivially perfect graphs as the comparability graphs of forests. Each component of a trivially perfect graph  $G$  has a model which is a rooted tree  $T$  with vertex set  $V(G)$ . Two vertices of  $G$  are adjacent if, in  $T$ , one lies on the path to the root of the other one. Thus each path from a leaf to the root is a maximal clique in  $G$  and these are all the maximal cliques. See [6,14] for the recognition of these graphs. In the following we assume that a rooted tree  $T$  as a model for the graph is a part of the (connected) input.

We simplify the problem by using two basic observations. (See [5,6] for similar observations.) Let  $T$  be a rooted tree which is the model for a connected trivially perfect graph  $G$ . Let  $R$  be the root of  $T$ ; note that this is a universal vertex in  $G$ . We assume that  $G$  is equipped with a  $\{k\}$ -assignment  $L$ , which attributes each vertex  $x$  with a pair  $(a_x, b_x)$  of numbers from  $\{0, \dots, k\}$ .

- (I) There exists a weak  $\{k\}$ -L-dominating function  $g$  of minimal cost such that

$$\forall_{x \in V(G) \setminus \{R\}} a_x > 0 \quad \Rightarrow \quad g(x) = a_x. \quad (23)$$

(II) There exists a weak  $\{k\}$ -L-dominating function  $g$  of minimal cost such that

$$\forall_{x \in V(G) \setminus \{R\}} a_x = 0 \quad \text{and} \quad b_x \leq \sum_{y \in N[x]} a_y \quad \Rightarrow \quad g(x) = 0. \quad (24)$$

**Definition 4.** The reduced instance of the weak  $\{k\}$ -L-domination problem is the subtree  $T'$  of  $T$  with vertex set  $V(G') \setminus W$ , where

$$W = \{ x \mid x \in V(G) \setminus \{R\} \text{ and } a_x > 0 \} \cup \{ x \mid x \in V(G) \setminus \{R\} \text{ and } a_x = 0 \text{ and } \sum_{y \in N[x]} a_y \geq b_x \}. \quad (25)$$

The labels of the reduced instance are, for  $x \neq R$ ,  $L(x) = (a'_x, b'_x)$ , where

$$a'_x = 0 \quad \text{and} \quad b'_x = b_x - \sum_{y \in N[x]} a_y, \quad (26)$$

and the root  $R$  has a label  $L(R) = (a'_R, b'_R)$ , where

$$a'_R = a_R \quad \text{and} \quad b'_R = \max \{ 0, b - \sum_{x \in V(G) \setminus \{R\}} a_x \}. \quad (27)$$

The previous observations prove the following lemma.

**Lemma 1.** Let  $T'$  and  $L'$  be a reduced instance of a weak  $\{k\}$ -L-domination problem. Then

$$\gamma_{wkL}(G) = \gamma_{wkL'}(G') + \sum_{x \in V(G) \setminus \{R\}} a_x.$$

In the following, let  $G$  be a connected, trivially perfect graph and let  $G$  be equipped with a  $\{k\}$ -assignment. Let  $G' = (V', E')$  be a reduced instance with model a  $T'$  and a root  $R$ , and a reduced assignment  $L'$ . Let  $g$  be a weak  $\{k\}$ - $L'$ -dominating function on  $G'$  of minimal cost. Notice that we may assume that

$$\boxed{\forall_{x \in V(G') \setminus \{R\}} g(x) \in \{0, 1\}.$$

Let  $x$  be an internal vertex in the tree  $T'$  and let  $Z$  be the set of descendants of  $x$ . Let  $P$  be the path in  $T'$  from  $x$  to the root  $R$ . Assume that  $Z$  is a union of  $r$  distinct cliques, say  $B_1, \dots, B_r$ . Assume that the vertices of each  $B_j$  are ordered  $x_1^j, \dots, x_{r_j}^j$  such that

$$\boxed{p \leq q \quad \Rightarrow \quad b'_{x_p^j} \geq b'_{x_q^j}.$$



Define  $d_{x_p^j} = b'_{x_p^j} - p + 1$ . Relabel the vertices of  $Z$  as  $z_1, \dots, z_\ell$  such that

$$\boxed{p \leq q \Rightarrow d_{z_p} \geq d_{z_q} .}$$

**Lemma 2.** *There exists an optimal weak  $\{k\}$ - $L'$ -dominating function  $g$  such that  $g(z_i) \geq g(z_j)$  when  $i < j$ .*

We moved the proof of this lemma to Appendix B.

**Definition 5.** *For  $a \in \{0, \dots, k\}$ ,  $a \geq a'_R$ , let  $\Gamma(G', L', a)$  be the minimal cost over all weak  $\{k\}$ - $L'$ -dominating functions  $g$  on  $G'$  on condition that  $g(P) \geq a$ .*

**Lemma 3.** *Define  $d_{z_{\ell+1}} = a$ . Let  $i^* \in \{1, \dots, \ell + 1\}$  be such that*

- (a)  $\max \{ a, d_{z_i^*} \} + i^* - 1$  *is smallest possible, and*
- (b)  $i^*$  *is smallest possible with respect to (a).*

*Let  $H = G' - Z$ . Let  $L^H$  be the restriction of  $L'$  to  $V(H)$  with the following modifications.*

$$\forall_{y \in P} b_y^H = \max \{ 0, b'_y - i^* + 1 \} .$$

*Let  $a^H = \max \{ a, d_{z_{i^*}} \}$ . Then*

$$\Gamma(G', L', a) = \Gamma(H, L^H, a^H) + i^* - 1 .$$

We moved the proof of this lemma to Appendix B.

The previous lemmas prove the following theorem.

**Theorem 5.** *Let  $G$  be a trivially perfect graph with  $n$  vertices. Let  $T$  be a rooted tree that represents  $G$ . Let  $k \in \mathbb{N}$  and let  $L$  be a  $\{k\}$ -assignment of  $G$ . Then there exists an  $O(k \cdot n)$  algorithm that computes a weak  $\{k\}$ - $L$ -dominating function of  $G$ .*

The related  $(j, k)$ -domination problem can be solved in linear time on trivially perfect graphs. The weak  $\{k\}$ - $L$ -domination problem can be solved in linear time on complete bipartite graphs. We moved that section to Appendix F.

## 4 2-Rainbow domination of interval graphs

In [4] the authors ask four questions, the last one of which is, whether there is a polynomial algorithm for the 2-rainbow domination problem on (proper) interval graphs. In this section we show that 2-rainbow domination can be solved in polynomial time on interval graphs.

We use the equivalence of the 2-rainbow domination problem with the weak  $\{2\}$ -domination problem. The equivalence of the two problems, when restricted to trees and interval graphs, was observed in [4]. Chang et al., proved that it holds for general  $k$  when restricted to the class of strongly chordal graphs [5]. The class of interval graphs is properly contained in that of the strongly chordal graphs.

An interval graph has a consecutive clique arrangement. That is a linear ordering  $[C_1, \dots, C_t]$  of the maximal cliques of the interval graph such that, for each vertex, the cliques that contain it occur consecutively in the ordering [13].

For a function  $g : V(G) \rightarrow \{0, 1, 2\}$  we write, as usual, for any  $S \subseteq V(G)$ ,

$$g(S) = \sum_{x \in S} g(x).$$

The weak  $\{2\}$ -domination problem is defined as follows.

**Definition 6.** *Let  $G$  be a graph. A function  $g : V(G) \rightarrow \{0, 1, 2\}$  is a weak  $\{2\}$ -dominating function on  $G$  if*

$$\forall_{x \in V(G)} \quad g(x) = 0 \quad \text{implies} \quad g(N[x]) \geq 2. \quad (28)$$

*The weak  $\{2\}$ -domination number of  $G$  is*

$$\gamma_{w2}(G) = \min \left\{ \sum_{x \in V(G)} g(x) \mid g \text{ is a weak } \{2\}\text{-domination function on } G \right\}. \quad (29)$$

Bršar and Šumenjak proved the following theorem [4].

**Theorem 6.** *When  $G$  is an interval graph,*

$$\gamma_{w2}(G) = \gamma_{r2}(G). \quad (30)$$

In the following, let  $G = (V, E)$  be an interval graph.

**Lemma 4.** *There exists a weak  $\{2\}$ -dominating function  $g$ , with  $g(V) = \gamma_{r2}(G)$ , such that every maximal clique has at most 2 vertices assigned the value 2.*

*Proof.* Assume that  $C_i$  is a maximal clique in the consecutive clique arrangement of  $G$ . Assume that  $C_i$  has 3 vertices  $x$ ,  $y$  and  $z$  with  $g(x) = g(y) = g(z) = 2$ . Assume that, among the three of them,  $x$  has the most neighbors in  $\cup_{j \geq i} C_j$  and that  $y$  has the most neighbors in  $\cup_{j \leq i} C_j$ . Then any neighbor of  $z$  is also a neighbor of  $x$  or it is a neighbor of  $y$ . So, if we redefine  $g(z) = 1$ , we obtain a weak  $\{2\}$ -dominating function with value less than  $g(V)$ , a contradiction.  $\square$

**Lemma 5.** *There exists a weak  $\{2\}$ -dominating function  $g$  with minimum value  $g(V) = \gamma_{r2}(G)$  such that every maximal clique has at most four vertices with value 1.*

*Proof.* The proof is similar to that of Lemma 4. Let  $C_i$  be a clique in the consecutive clique arrangement of  $G$ . Assume that  $C_i$  has 5 vertices  $x_i$ ,  $i \in \{1, \dots, 5\}$ , with  $g(x_i) = 1$  for each  $i$ . Order the vertices  $x_i$  according to their neighborhoods in  $\cup_{j \geq i} C_j$  and according to their neighborhoods in  $\cup_{j \leq i} C_j$ . For simplicity, assume that  $x_1$  and  $x_2$  have the most neighborhoods in the first union of cliques and that  $x_3$  and  $x_4$  have the most neighbors in the second union of cliques. Then  $g(x_5)$  can be reduced to zero; any other vertex that has  $x_5$  in its neighborhood already has two other 1's in it.

This proves the lemma.  $\square$

**Theorem 7.** *There exists a polynomial algorithm to compute the 2-rainbow domination number for interval graphs.*

We moved the proof of this theorem and some remarks to Appendix D.

We obtained similar results for the class of permutation graphs. We moved that section to Appendix E.

## 5 NP-Completeness for splitgraphs

A graph  $G$  is a splitgraph if  $G$  and  $\bar{G}$  are both chordal. A splitgraph has a partition of its vertices into two set  $C$  and  $I$ , such that the subgraph induced by  $C$  is a clique and the subgraph induced by  $I$  is an independent set.

Although the NP-completeness of  $k$ -rainbow domination for chordal graphs was established in [6], their proof does not imply the intractability for the class of splitgraphs. However, that is easy to mend.

**Theorem 8.** *Let  $k \in \mathbb{N}$ . Computing  $\gamma_{rk}(G)$  is NP-complete for splitgraphs.*

We moved the proof of this theorem to Appendix C.

Similarly, we have the following theorem.

**Theorem 9.** *Let  $k \in \mathbb{N}$ . Computing  $\gamma_{wk}(G)$  is NP-complete for splitgraphs.*

We moved the proof of this theorem to Appendix C.

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## A Rainbow domination on $P_4$ -sparse graphs

Many problems can be solved in linear time for the class of cographs and  $P_4$ -sparse graphs, see eg, [10, Theorem 2 and Corollary 3]. Both classes are of bounded cliquewidth (or rankwidth). We are not aware of many problems of which the complexities differ on the two classes of graphs. An interesting example, that might have different complexities on cographs and  $P_4$ -sparse graphs, is the weighted coloring problem [2]. In this section we show that the  $k$ -rainbow domination problem can be solved in linear time on  $P_4$ -sparse graphs.

Hoàng introduced  $P_4$ -sparse graphs as follows.

**Definition 7.** *A graph is  $P_4$ -sparse if every set of 5 vertices contains at most one  $P_4$  as an induced subgraph.*

Jamison and Olariu showed that  $P_4$ -sparse graphs have a decomposition similar to the decomposition of cographs. To define it we need the concept of a spider.

**Definition 8.** *A graph  $G$  is a thin spider if its vertices can be partitioned into three sets  $S$ ,  $K$  and  $T$ , such that*

(a)  *$S$  induces an independent set and  $K$  induces a clique and*

$$|S| = |K| \geq 2.$$

(b) *Every vertex of  $T$  is adjacent to every vertex of  $K$  and to no vertex of  $S$ .*

(c) *There is a bijection from  $S$  to  $K$ , such that every vertex of  $S$  is adjacent to exactly one unique vertex of  $K$ .*

*A thick spider is the complement of a thin spider.*

Notice that, possibly,  $T = \emptyset$ . The set  $T$  is called the head, and  $K$  and  $S$  are called the body and the feet of the thin spider.

Jamison and Olariu proved the following theorem [18].

**Theorem 10.** *A graph is  $P_4$ -sparse if and only if for every induced subgraph  $H$  one of the following holds.*

- (i)  *$H$  is disconnected, or*
- (ii)  *$\bar{H}$  is disconnected, or*
- (iii)  *$H$  is isomorphic to a spider.*

**Theorem 11.** *There exists a linear-time algorithm to compute the  $k$ -rainbow domination number for  $P_4$ -sparse graphs and  $k \in \mathbb{N}$ .*

*Proof.* We extend Formula (12) in the proof of Theorem 2 to include spiders.

Assume that  $G$  is a thin spider, with a head  $T$ , a body  $K$  and an independent set of feet  $S$ . We need to consider only  $k$ -rainbow colorings such that at least one vertex of  $G$  has an empty label. Notice that we may assume that all the feet have labels of cardinality at most one. Furthermore, there is at most one vertex in  $S$  that has an empty label.

First assume that one foot, say  $x$ , has an empty label. Then its neighbor, say  $y \in K$  has label  $[k]$ . In that case, the (only) optimal  $k$ -rainbow coloring has

- (a) all vertices in  $S \setminus \{x\}$  a label of cardinality 1,
- (b) all vertices in  $K \setminus \{y\}$  an empty label, and
- (c) all vertices of  $T$  also an empty label.

It follows that the cost in this case is

$$|S| - 1 + k. \quad (31)$$

All other optimal  $k$ -rainbow colorings give all feet a label of cardinality 1. If some vertex of  $T$  has an empty label, then its neighborhood has cost at least  $k$ , and the total cost will be more than (31). So, the only alternative  $k$ -rainbow coloring assigns no empty label to  $S$  nor  $T$ , and assigns an empty label to some vertex  $v$  of  $K$ . In such a case, the neighborhood of  $v$  has cost at least  $k$ , so that combining with the cost in the non-neighboring feet, the total cost is at least that of (31).

In conclusion, the optimal cost of a thin spider  $G$  is

$$\min \{ |V(G)|, |S| - 1 + k \}. \quad (32)$$

The case analysis for the thick spider is similar. This proves the theorem.  $\square$

*Remark 3.* A graph  $G$  is  $P_4$ -lite if every induced subgraph  $H$  of at most 6 vertices satisfies one of the following.

- (i)  $H$  contains at most two induced  $P_4$ 's, or
- (ii)  $H$  is isomorphic to  $S_3$  or  $\bar{S}_3$ .

These graphs contain the  $P_4$ -sparse graphs.

Babel and Olariu studied  $(q, q-4)$ -graphs, see, eg, [3]. It would be interesting to study the complexity of the  $k$ -rainbow problem on these graphs.

## B Trivially perfect graphs

**Lemma 6.** *There exists an optimal weak  $\{k\}$ - $L'$ -dominating function  $g$  such that  $g(z_i) \geq g(z_j)$  when  $i < j$ .*

*Proof.* Assume not. Let  $\hat{g}$  be the same as  $g$  except that  $\hat{g}(z_i) = 1$  and  $\hat{g}(z_j) = 0$ . To see that  $\hat{g}$  is a weak  $\{k\}$ - $L$ -dominating function, first notice that, when  $z_i$  and  $z_j$  are in a common clique of  $G'[Z]$  then it is easy to see that  $\hat{g}$  defines a weak  $\{k\}$ - $L$ -dominating function of at most the same cost.

Now assume that the claim holds for all pairs contained in common cliques. Then choose  $z_i$  and  $z_j$  with  $i < j$ ,  $g(z_i) = 0$  and  $g(z_j) = 1$ , and  $z_i$  and  $z_j$  in different cliques with  $i$  as small as possible and  $j$  as large as possible. Say  $z_i = x_{m^*}^m$  and  $z_j = x_{h^*}^h$ . Then, by assumption,

$$\forall_{t < m^*} g(x_t^m) = 1 \quad \forall_{t > m^*} g(x_t^m) = 0 \quad \forall_{t < h^*} g(x_t^h) = 1 \quad \forall_{t > h^*} g(x_t^h) = 0.$$

Now, notice that

$$g(N(z_i)) = m^* - 1 + g(P) \geq b'_{z_i} \quad \Rightarrow \quad g(P) \geq b'_{z_i} - m^* + 1.$$

We prove that  $\hat{g}(N[z_j]) \geq b'_{z_j}$ . We have that

$$\hat{g}(N[z_j]) = g(P) + h^* - 1 \geq b'_{z_i} - m^* + h^*.$$

By definition and the ordering on  $Z$ ,

$$b'_{z_i} - m^* + 1 = d_{z_i} \geq d_{z_j} = b'_{z_j} - h^* + 1 \quad \Rightarrow \quad \hat{g}(N[z_j]) \geq b'_{z_j}.$$

This proves the lemma.  $\square$

**Lemma 7.** *Define  $d_{z_{\ell+1}} = a$ . Let  $i^* \in \{1, \dots, \ell + 1\}$  be such that*

- (a)  $\max \{ a, d_{z_i^*} \} + i^* - 1$  *is smallest possible, and*
- (b)  $i^*$  *is smallest possible with respect to (a).*

*Let  $H = G' - Z$ . Let  $L^H$  be the restriction of  $L'$  to  $V(H)$  with the following modifications.*

$$\forall_{y \in P} b_y^H = \max \{ 0, b'_y - i^* + 1 \}.$$

*Let  $a^H = \max \{ a, d_{z_{i^*}} \}$ . Then*

$$\Gamma(G', L', a) = \Gamma(H, L^H, a^H) + i^* - 1.$$

*Proof.* Let  $g$  be a weak  $\{k\}$ - $L'$ -dominating function, satisfying  $g(P) \geq a$ , of minimal cost. We first prove that the restriction of  $g$  to  $H$  is a weak  $\{k\}$ - $L^H$ -dominating function and that  $g(P) \geq a^H$ .



Let  $i$  be the largest index such that  $g(z_j) = 1$  for all  $j < i$ . If  $i = \ell + 1$  we have  $g(P) \geq a = d_{z_{\ell+1}}$ .

Now assume that  $i \leq \ell$ . Let  $z_i = x_{m^*}^m$ . Then

$$g(N[z_i]) = g(P) + m^* - 1 \geq b'_{z_i} \Rightarrow g(P) \geq b'_{z_i} - m^* + 1 = d_{z_i}.$$

Thus  $g(P) \geq \max\{a, d_{z_i}\}$  and so we have that

$$g(P) + i - 1 \geq \max\{a, d_{z_i}\} + i - 1 \geq \max\{a, d_{z_{i^*}}\} + i^* - 1.$$

We claim that  $i^* \geq i$ .

Suppose that  $i^* < i$ . Then let  $\hat{g}$  be the same as  $g$  except that

$$\hat{g}(R) = \min\{g(R) + i - i^*, k\} \quad \text{and} \quad \forall i^* \leq j < i \quad \hat{g}(z_j) = 0.$$

Let  $j \geq i^*$  and let  $h$  and  $h^*$  be such that  $z_j = x_{h^*}^h$ . Let  $t$  be the smallest index for which  $\hat{g}(x_t^h) = 0$ . By the inequality above,  $\hat{g}(P) \geq \max\{a, d_{z_{i^*}}\}$ , and so,

$$\hat{g}(N[x_{h^*}^h]) = \hat{g}(N[x_t^h]) = \hat{g}(P) + t - 1 \geq d_{z_{i^*}} + t - 1 \geq d_{x_t^h} + t - 1 = b'_{x_t^h} \geq b'_{x_{h^*}^h}.$$

Thus  $\hat{g}$  is a weak  $\{k\}$ - $L'$ -dominating function on  $G'$  with  $\hat{g}(P) \geq a$  and with cost at most  $\|g\|$ . Therefore, we may assume that

$$i^* \geq i \quad \text{and} \quad g(P) \geq \max\{a, d_{z_{i^*}}\} = a^H.$$

Also, notice that

$$\begin{aligned} \forall y \in P \quad g(y) = 0 &\Rightarrow \\ g^H(N[y]) \geq \max\{0, g(N[y]) - i + 1\} &\geq \max\{0, b'_y - i^* + 1\} = b_y^H. \end{aligned} \quad (33)$$

This proves that

$$\Gamma(G', L', a) - i^* + 1 \geq \Gamma(H, L^H, a^H).$$

Now let  $g^H$  be a weak  $\{k\}$ - $L^H$ -dominating function of  $H$ , with  $g^H(P) \geq a^H$ , of minimal cost. Extend  $g^H$  to a function  $g'$  on  $G'$  by defining  $g'(z_j) = 1$  for all  $j < i^*$  and  $g'(z_j) = 0$  for all  $j \geq i^*$ . We claim that  $g'$  is a weak  $\{k\}$ - $L$ -dominating function of  $G'$ . Let  $i \geq i^*$ . Say  $z_i = x_{m^*}^m$ . Let  $m^*$  be the first index such that  $g'(x_j^m) = 0$  for  $j \geq m^*$ . Then

$$g'(N[z_i]) = g'(P) + m^* - 1 \geq d_{z_{i^*}} + m^* - 1 \geq d_{z_i} + m^* - 1 = b_{z_i}.$$

For  $i < i^*$  we have  $g'(z_i) \neq 0$ . For the vertices  $y \in P$ ,  $g'(N[y]) = k$  or

$$g'(N[y]) \geq g^H(N_H[y]) + i^* - 1 \geq b_y^H + i^* - 1 \geq b'_y.$$

This proves the lemma.  $\square$

## C NP-Completeness proofs for splitgraphs

**Theorem 12.** *For each  $k \in \mathbb{N}$ , the  $k$ -rainbow domination problem is NP-complete for splitgraphs.*

*Proof.* Assume that  $G$  is a splitgraph with maximal clique  $C$  and independent set  $I$ . Construct an auxiliary graph  $G'$  by making  $k-1$  pendant vertices adjacent to each vertex of  $C$ . Thus  $G'$  has  $|V(G)| + |C|(k-1)$  vertices, and  $G'$  remains a splitgraph. We prove that

$$\gamma_{rk}(G') = \gamma(G) + |C| \cdot (k-1).$$

Since domination is NP-complete for splitgraphs [4], this proves that  $k$ -rainbow domination is NP-complete also.

We first show that

$$\gamma_{rk}(G') \leq \gamma(G) + |C| \cdot (k-1).$$

Consider a dominating set  $D$  of  $G$  with  $|D| = \gamma(G)$ . We use  $D$  to construct a  $k$ -rainbow function  $f$  for  $G'$  as follows:

- For any  $v \in D$ , if  $v \in C$ , let  $f(v) = [k]$ ; else, if  $v \in I$ , let  $f(v) = \{k\}$ ;
- For any  $v \in V(G) \setminus D$ , let  $f(v) = \emptyset$ ;
- For the  $k-1$  pendant vertices attaching to a vertex  $v \in C$ , if  $f(v) = [k]$ , then  $f$  assigns to each of these pendant vertices an empty set. Otherwise, if  $f(v) = \emptyset$ , then  $f$  assigns the distinct size-1 sets  $\{1\}, \{2\}, \dots, \{k-1\}$  to these pendant vertices, respectively.

It is straightforward to check that  $f$  is a  $k$ -rainbow function. Moreover, we have

$$\gamma_{rk}(G') \leq \sum_{x \in V(G')} |f(x)| = \gamma(G) + |C| \cdot (k-1). \quad (34)$$

We now show that

$$\gamma_{rk}(G') \geq \gamma(G) + |C| \cdot (k-1).$$

Consider a minimizing  $k$ -rainbow function  $f$  for  $G'$ . Without loss of generality, we further assume that  $f$  assigns either  $\emptyset$  or a size-1 subset to each pendant vertex.<sup>3</sup> Define  $D \subseteq V(G)$  as

$$D = \{ x \mid f(x) \neq \emptyset \text{ and } x \in V(G) \}. \quad (35)$$

That is,  $D$  is formed by removing all the pendant vertices in  $G'$ , and selecting all those vertices where  $f$  assigns a non-empty set. Observe that  $D$  is a dominating set of  $G$ .<sup>4</sup> Moreover, we have

<sup>3</sup> Otherwise, if a pendant vertex  $p$  attaching  $v$  is assigned a set with two or more labels, say  $f(p) = \{\ell_1, \ell_2, \dots\}$ , we modify  $f$  into  $f'$  so that  $f'(p) = \{\ell_1\}$ ,  $f'(v) = f(v) \cup (f(p) \setminus \{\ell_1\})$ , and  $f'(x) = f(x)$  for the remaining vertices; the resulting  $f'$  is still a minimizing  $k$ -rainbow function.

<sup>4</sup> That is so because for any  $v \in V(G) \setminus D$ , we have  $f(v) = \emptyset$  so that the union of labels of  $v$ 's neighbor in  $G'$  is  $[k]$ ; however, at most  $k-1$  neighbors of  $v$  are removed, and each was assigned a size-1 set, so that  $v$  must have at least one neighbor in  $D$ .

$$\begin{aligned}
|D| &= \sum_{x \in C} [f(x) \neq \emptyset] + \sum_{x \in I} [f(x) \neq \emptyset] \\
&\leq \sum_{x \in V(G') \setminus I} |f(x)| - |C| \cdot (k-1) + \sum_{x \in I} |f(x)| \\
&\leq \sum_{x \in V(G')} |f(x)| - |C| \cdot (k-1),
\end{aligned}$$

where the first inequality follows from the fact that for each  $v \in C$  and its corresponding pendant vertices  $P_v$ ,

$$|f(v)| + \sum_{x \in P_v} |f(x)| - (k-1) = \begin{cases} 0 & \text{if } f(v) = \emptyset \\ \geq 1 & \text{if } f(v) \neq \emptyset. \end{cases}$$

Consequently, we have

$$\gamma(G) \leq |D| \leq \gamma_{rk}(G') - |C| \cdot (k-1). \quad (36)$$

This proves the theorem.  $\square$

**Theorem 13.** *For each  $k \in \mathbb{N}$ , the weak  $\{k\}$ -domination problem is NP-complete for splitgraphs.*

*Proof.* Let  $G$  be a splitgraph with maximal clique  $C$  and independent set  $I$ . Construct the graph  $G'$  as in Theorem 8, by adding  $k-1$  pendant vertices to each vertex of the maximal clique  $C$ . We prove that

$$\gamma_{wk}(G') = \gamma(G) + |C| \cdot (k-1).$$

First, let us prove that

$$\gamma_{wk}(G') \leq \gamma(G) + |C|(k-1).$$

Let  $D$  be a minimum dominating set. Construct a weak  $\{k\}$ -domination function  $g : V(G') \rightarrow \{0, \dots, k\}$  as follows.

- (i) For  $x \in D \cap C$ , let  $g(x) = k$ .
- (ii) For  $x \in D \cap I$ , let  $g(x) = 1$ .
- (iii) For  $x \in V(G) \setminus D$ , let  $g(x) = 0$ .
- (iv) For a pendant vertex  $x$  with  $N(x) \in D$ , let  $g(x) = 0$ .
- (v) For a pendant vertex  $x$  with  $N(x) \notin D$ , let  $g(x) = 1$ .

It is easy to check that  $g$  is a weak  $\{k\}$ -dominating function with cost

$$\gamma_{wk}(G') \leq \sum_{x \in V(G')} g(x) = \gamma(G) + |C| \cdot (k-1).$$

To prove the converse, let  $g$  be a weak  $\{k\}$ -dominating function for  $G'$  of minimal cost. We may assume that  $g(x) \in \{0, 1\}$  for every pendant vertex  $x$ . Define

$$D = \{ x \mid x \in V(G) \text{ and } g(x) > 0 \}.$$

Then  $D$  is a dominating set of  $G$ . Furthermore,

$$\begin{aligned} \gamma(G) &\leq |D| = \sum_{x \in C} [g(x) > 0] + \sum_{x \in I} [g(x) > 0] \\ &\leq \sum_{x \in V(G') \setminus I} g(x) - |C| \cdot (k-1) + \sum_{x \in I} g(x) \\ &\leq \sum_{x \in V(G')} g(x) - |C| \cdot (k-1) \\ &\leq \gamma_{wk}(G') - |C| \cdot (k-1). \end{aligned}$$

This proves the theorem.  $\square$

## D 2-Rainbow domination of interval graphs

**Theorem 14.** *There exists a polynomial algorithm to compute the 2-rainbow domination number for interval graphs.*

*Proof.* By Lemmas 4 and 5 there is a polynomial dynamic programming algorithm which solves the problem. Let  $[C_1, \dots, C_t]$  be a consecutive clique arrangement. For each  $i$  the algorithm computes a table of partial 2-rainbow domination numbers for the subgraph induced by  $\cup_{\ell=1}^i C_\ell$ , parameterized by a given subset of vertices in  $C_i$  that are assigned the values 0, 1 and 2. We say that a vertex  $x \in C_i$  is *satiated* if

$$g(N(x) \cap (\cup_{\ell=1}^i C_\ell)) \geq 2.$$

The tabulated rainbow domination numbers are partial in the sense that the neighborhood condition is not necessarily satisfied for all vertices in  $C_i$  that are assigned the value 0. The vertices of  $C_i$  that are assigned the value 0 which are not satiated, either need an extra 1 or an extra 2. Each of the two subsets of nonsatiated vertices is characterized by one representative vertex, the one among them that extends furthest to the left.

In total, the dynamic system is characterized by 4 state variables. They are

- (i) the set of, at most two 2's,
- (ii) the set of, at most four 1's,

- (iii) the nonsatiated vertex that extends furthest to the left and that needs an extra 1, and
- (iv) a similar nonsatiated vertex that needs an extra 2.

Each clique has at most  $n = |V(G)|$  vertices and so, there are at most  $n^2 \cdot n^4 \cdot n \cdot n = n^8$  different assignments of the state variables. In the transition  $i \rightarrow i + 1$ , the table is computed by minimizing, for all sensible assignments of vertices in  $C_{i+1}$ , over the compatible values in the table at stage  $i$ . Each update takes  $O(1)$  time. The number of cliques is at most  $n$ , and so the algorithm runs in  $O(n^9)$  time.  $\square$

*Remark 4.* The observations above can be generalized to show that, for each  $k$ , the  $k$ -rainbow domination problem can be solved in polynomial time for interval graphs. However, this leaves open the question whether  $k$ -rainbow domination is in FTP for interval graphs.

*Remark 5.* Let  $A$  be the closed neighborhood matrix of an interval graph or a strongly chordal graph  $G$ . Then  $A$  is totally balanced, that is, after a suitable permutation of rows and columns the matrix becomes greedy, ie,  $\Gamma$ -free. A matrix is greedy if it has no  $2 \times 2$  submatrix  $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ . For (totally) balanced matrices some polyhedra have only integer extreme points. This leads to polynomial algorithms for domination on eg, strongly chordal graphs. Notice that the closed neighborhood matrix of  $G \square K_2$  is

$$B = \begin{pmatrix} A & I \\ I & A \end{pmatrix}. \quad (37)$$

To solve the 2-rainbow domination problem for strongly chordal graphs, one needs to solve the following integer programming problem.

$$\min \sum_{i=1}^{2n} x_i \quad \text{such that} \quad (38)$$

$$B\mathbf{x} \geq \mathbf{1} \quad \text{and} \quad \forall_i x_i \in \{0, 1\}. \quad (39)$$

For the 2-rainbow domination problem it would be interesting to know on what conditions on the adjacency matrix  $A$ , the matrix  $B$  is (totally) balanced.

## E 2-Rainbow domination of permutation graphs

Permutation graphs properly contain the class of cographs. They can be defined as the intersection graphs of a set of straight line segments that have their endpoints on two parallel lines. In other words, a graph  $G$  is a permutation graph whenever both  $G$  and  $\bar{G}$  are comparability graphs [9,19].

We can use a technique similar to that used for interval graphs to show that 2-rainbow domination is polynomial for permutation graphs.

An intersection model of straight line segments for a permutation graph, as described above, is called a permutation diagram.

**Definition 9.** *Consider a permutation diagram for a permutation graph  $G$ . A scanline is a straight line segment with endpoints on the two parallel lines of which neither of the endpoints coincides with any endpoint of a line segment of the permutation diagram.*

We proceed as in Section 4. Consider a 2-rainbow function  $f$  for a graph  $G = (V, E)$ . For a set  $S \subseteq V$  we write,  $f(S) = \cup_{x \in S} f(x)$ .

**Lemma 8.** *Let  $G = (V, E)$  be a permutation graph and let  $f$  be a 2-rainbow function with  $\sum |f(x)| = \gamma_{r2}(G)$ . Consider a permutation diagram with two parallel horizontal lines. Let  $s$  be a scanline. There are at most 4 line segments that cross  $s$  with function value  $\{1, 2\}$ .*

*Proof.* Assume that there are at least 5 such line segments. Take 4 of them, each one with the endpoint as far as possible from the endpoint of  $s$  on the top and bottom line. Any line segment crossing the fifth, crosses also at least one of these four. This shows that the function value of the fifth line can be replaced by  $\emptyset$ . This contradicts the optimality of  $f$ .

This proves the lemma.  $\square$

**Lemma 9.** *For every scanline  $s$  there are at most 8 line segments that cross it and that have function values not  $\emptyset$ .*

*Proof.* The proof is similar to the proof of Lemma 8. Consider the 8 line segments that cross  $s$ , and with function values that are nonempty subsets of  $\{1, 2\}$ , and of which the union is pairwise  $\{1, 2\}$ , and that are furthest away from the endpoints of  $s$ .

Any other line segment crossing  $s$  with function value  $\neq \emptyset$  would contradict the optimality of  $f$ .  $\square$

**Theorem 15.** *There exists a polynomial algorithm to compute the 2-rainbow domination number for permutation graphs.*

*Proof.* Consider a dynamic programming algorithm, similar to that described for interval graphs in Theorem 7, that proceeds by moving a scanline from left to right through the permutation diagram.  $\square$

*Remark 6.* Obviously, a similar technique shows that the weak  $\{2\}$ -domination number can be computed in polynomial-time for permutation graphs. Perhaps it is interesting to ask the question whether these two domination numbers are equal for the class of permutation graphs.

## F Weak $\{k\}$ -L-domination on complete bipartite graphs

For integers  $1 \leq j \leq k$ , a  $(j, k)$ -dominating function on a graph  $G$  is a function  $g : V(G) \rightarrow \{0, \dots, j\}$  such that for every vertex  $x$ ,  $g(N[x]) \geq k$  [19,20]. The  $(j, k)$ -domination number  $\gamma_{j,k}(G)$  is the minimal cost of a  $(j, k)$ -dominating function.

**Theorem 16.** *Let  $G$  be trivially perfect. There exists a linear-time algorithm to compute  $\gamma_{j,k}(G)$ .*

*Proof.* Let  $T$  be a tree model for  $G$ . It is easy to check that the following procedure solves the problem. Color the vertices of the first  $\lfloor \frac{k}{j} \rfloor$  BFS-levels of the tree  $T$  with color  $j$ , and color the vertices in the next level with the remainder  $k - \lfloor \frac{k}{j} \rfloor \cdot j = k \bmod j$ .  $\square$

We show that the weak  $\{k\}$ -L-domination problem can be solved in linear time on complete bipartite graphs. Let  $G$  be complete bipartite, with color classes  $V$  and  $V'$ . Let  $L$  be a  $\{k\}$ -assignment, that is,  $L$  assigns a pair  $(a_x, b_x)$  of numbers from  $\{0, \dots, k\}$  to every vertex  $x$ . For simplicity we may assume that, for all vertices  $x$ ,

$$a_x = 0.$$

For simplicity we also assume that  $|V| = |V'| = n$ . Denote the  $b$ -labels of vertices in  $V$  by  $b(1), \dots, b(n)$  and denote the  $b$ -labels of vertices in  $V'$  by  $b'(1), \dots, b'(n)$ . We assume that these are ordered such that

$$b(1) \geq \dots \geq b(n) \quad \text{and} \quad b'(1) \geq \dots \geq b'(n).$$

Then the weak  $\{k\}$ -L-domination problem can be formulated as follows. Let  $b(n+1) = b'(n+1) = 0$ .

$$\begin{aligned} & \min x + y \\ & \text{subject to } x \geq b'(y+1) \quad \text{and} \quad y \geq b(x+1). \end{aligned}$$

**Theorem 17.** *The weak  $\{k\}$ -L-domination problem can be solved in linear time on complete bipartite graphs.*

*Proof.* Let, for  $x \in \{0, \dots, n\}$ ,

$$\begin{aligned} y^1(x) &= \min \{ y \mid y \in \{0, \dots, n\} \quad \text{and} \quad y \geq b(x+1) \} = b(x+1) \\ y^2(x) &= \min \{ y \mid y \in \{0, \dots, n\} \quad \text{and} \quad x \geq b'(y+1) \}. \end{aligned}$$

Let

$$m(x) = \max \{ y^1(x), y^2(x) \}.$$

Then the solution to the weak  $\{k\}$ -L-domination problem is

$$\min \{ x + m(x) \mid x \in \{0, \dots, n\} \}.$$

It is easy to check that the values  $y^1(x)$  and  $y^2(x)$  can be computed in, overall, linear time. This proves the theorem.  $\square$

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